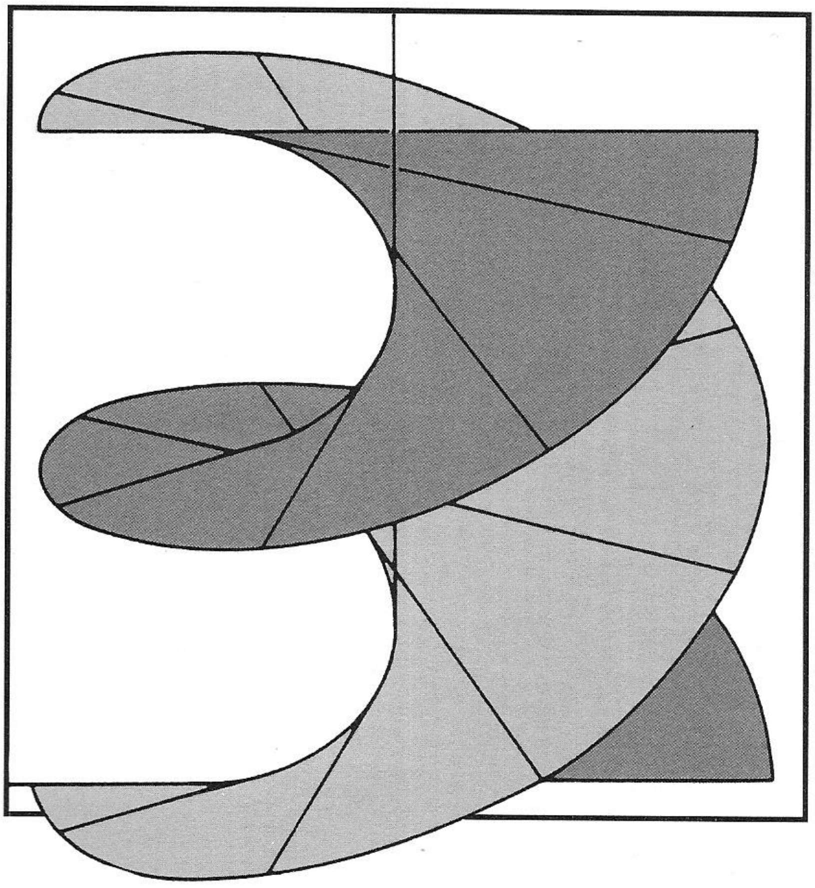
M434 Part V
Mathematics: A Fourth Level Course

# M434
# DIFFERENTIAL GEOMETRY



## PART V
## SHAPE OPERATORS

The Open University

The Open University

Mathematics: A Fourth Level Course

# M434 Differential Geometry

## Part V Shape Operators

Prepared for the Course Team
by Bob Margolis

## Set book

Barrett O'Neill, *Elementary Differential Geometry*, hardback edition (Academic Press, 1966).

It is essential to have this book; the course is based on it and will not make sense without it.

The set book is referred to as *O'Neill*.

The Open University, Walton Hall, Milton Keynes, MK7 6AA.

First published as *M334* 1976.

First published as *M434* 1993. Reprinted 1996.

Copyright © 1993 The Open University

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means, without written permission from the publisher or a licence from the Copyright Licensing Agency Limited. Details of such licences (for reprographic reproduction) may be obtained from the Copyright Licensing Agency Ltd of 90 Tottenham Court Road, London, W1P 9HE.

Edited, designed and typeset by the Open University using the Open University TEX System.

Printed in the United Kingdom by Hobbs the Printers Ltd, Totton, Hampshire.

ISBN 0 7492 4774 6

This text forms part of an Open University Fourth Level Course. If you would like a copy of *Studying with The Open University*, please write to the Central Enquiry Service, PO Box 200, The Open University, Walton Hall, Milton Keynes, MK7 6YZ. If you have not already enrolled on the Course and would like to buy this or other Open University material, please write to Open University Educational Enterprises Ltd, 12 Cofferidge Close, Stony Stratford, Milton Keynes, MK11 1BY, United Kingdom.

1.2

# Contents

Introduction 4

1 The shape operator 5

2 Normal curvature 9

3 Gaussian curvature 13

4 Computational techniques 16

5 Special curves 20

6 Surfaces of revolution 23

7 Summary 24

Solutions to the exercises 25

# Introduction

Having developed the computational tools required, we can now apply them to the study of the 'shape' of surfaces in $\mathbf{E}^3$.

Section 1 is concerned with a definition of 'shape' and makes use of the idea of unit normal vector fields.

The idea behind using normal fields is, we think, a fairly natural one. Imagine living on the surface of a cylinder and carrying a unit normal with you from place to place. If you move along one of the generating lines then the unit normal will always point in the same direction—its rate of change is zero. However, moving along one of the cross-section circles causes the unit normal to rotate steadily.

That is, parallel to the axis of the cylinder.

The rate of change of the normal measures, in some sense, the shape of the surface along the path that you follow on the surface.

Section 2 makes use of the definitions of Section 1 to define curvature for surfaces. Since, as the cylinder example suggests, you would expect to experience different curvatures when moving in different directions, the curvature is a function not only of position on the surface, but also of direction. This section also provides some computational techniques.

Section 3 introduces two measurements of curvature that summarize the shape of a surface at each point. They are functions and, therefore, lose the information about differences in curvature in different directions. However, they are useful general descriptions.

The main purpose of Section 4 is to provide a range of computational techniques based on the partial velocities and their derivatives. In a sense the formulas derived correspond to those for curves that were expressed in terms of $\alpha'$, $\alpha''$ and $\alpha'''$.

The functions

$$E = \mathbf{x}_u \cdot \mathbf{x}_u, \quad F = \mathbf{x}_u \cdot \mathbf{x}_v \quad \text{and} \quad G = \mathbf{x}_v \cdot \mathbf{x}_v,$$

which you have already met, play a central role in the computations.

Sections 5 and 6 apply the machinery that has been developed to two topics. The first is the discussion of particular types of curves in surfaces; the second is to surfaces of revolution.

Finally, Section 7 provides the usual summary.

# Study advice

The following represents a possible plan for study weeks.

**Week 1** *O'Neill*, Chapter V, Sections 1 and 2.
**Week 2** *O'Neill*, Chapter V, Section 3 and TMA03.
**Week 3** *O'Neill*, Chapter V, Sections 4 and 5.
**Week 4** *O'Neill*, Chapter V, Section 6.

This leaves two study weeks for the work on three sections of Chapter VI and TMA04. Since the fourth week listed above is probably a little light, you may wish to start work on TMA04 during that week.

# 1 The shape operator

---

**Read** *O'Neill, Chapter V, Section 1, pages 189–193.*

---

The fundamental purpose of this section is to define how 'shape' is measured for *connected* surfaces in $\mathbf{E}^3$.

Connectedness is required so that the number of unit normal vector fields is limited. On the image of any patch there is always a choice of two unit normal vector fields: one in the same direction as

$$\mathbf{x}_u \times \mathbf{x}_v$$

at each point and the other in the opposite direction.

If a surface is orientable then there is a non-vanishing normal vector field which can be made unit length. The resulting unit normal, together with its negative, gives a choice of two unit normals on each connected component of the surface. If the surface has $n$ connected components, then we can define $2^n$ unit normal vector fields.

We proved this result earlier.

Limiting ourselves to connected, orientable surfaces simplifies things somewhat. The theory developed can then be applied to each connected component of more general surfaces.

***Covariant derivative*** *O'Neill* does not give a definition as such, although a definition is hinted at on page 189. The definition is

***Definition 1.0***

Let $V$ be a vector field defined on a surface $M \subset \mathbf{E}^3$ and let $\mathbf{v_p}$ be a vector tangent to $M$. Then the covariant derivative $\nabla_{\mathbf{v_p}} V$ is defined as

$$V(\alpha(t))'(0),$$

where $\alpha$ is any curve in $M$ such that

$$\alpha(0) = \mathbf{p},\ \alpha'(0) = \mathbf{v}.$$

This definition corresponds exactly to the definition of directional derivative on a surface.

Because of the importance of the parameter curves, we shall look at covariant derivatives with respect to the partial velocities.

The partial velocities form a basis for the tangent space at each point.

Suppose, then, that $M \subset \mathbf{E}^3$ is a surface and that $\mathbf{x}$ is a parametrization of $M$ (or part of it). Let $V$ be a vector field defined on the image of $\mathbf{x}$. We want to calculate

$$\nabla_{\mathbf{x}_u} V \quad \text{and} \quad \nabla_{\mathbf{x}_v} V.$$

We tackle them in turn.

The definition requires a curve with velocity $\mathbf{x}_u$, the obvious one to take is the $u$-parameter curve. To be quite explicit, we shall calculate

$$\nabla_{\mathbf{x}_u} V$$

at the point

$$\mathbf{p} = \mathbf{x}(u_0, v_0).$$

We define $\alpha$ by

$$\alpha(t) = \mathbf{x}(u_0 + t, v_0).$$

Since

$$\begin{aligned}\alpha'(t) &= \mathbf{x}_u(u_0+t,v_0)\times 1+\mathbf{x}_v(u_0+t,v_0)\times 0 \quad \text{(chain rule)}\\ &= \mathbf{x}_u(u_0+t,v_0),\end{aligned}$$

we have

$$\alpha'(0)=\mathbf{x}_u(u_0,v_0) \quad \text{and also} \quad \alpha(0)=\mathbf{x}(u_0,v_0)=\mathbf{p}.$$

Thus $\alpha$ is a suitable curve to use in the definition of covariant derivative. The definition then gives

$$\nabla_{\mathbf{x}_u(u_0,v_0)}V = V(\alpha(t))'(0).$$

Now $V(\alpha(t)) = V(\mathbf{x}(u_0+t,v_0))$ and the chain rule then gives

$$\begin{aligned}V(\alpha(t))' &= \frac{\partial V(\mathbf{x}(u,v))}{\partial u}(u_0+t,v_0)\times 1+\frac{\partial V(\mathbf{x}(u,v))}{\partial v}(u_0+t,v_0)\times 0\\ &= \frac{\partial V(\mathbf{x})}{\partial u}(u_0+t,v_0).\end{aligned}$$

Evaluating at $t=0$ shows that

$$\nabla_{\mathbf{x}_u(u_0,v_0)}V = \frac{\partial V(\mathbf{x}(u,v))}{\partial u}(u_0,v_0).$$

If we express this result in 'function' form, we have

$$\nabla_{\mathbf{x}_u}V=\frac{\partial V(\mathbf{x})}{\partial u}.$$

Exactly similar arguments show that

$$\nabla_{\mathbf{x}_v}V=\frac{\partial V(\mathbf{x})}{\partial v}.$$

These two results are of immense importance for doing computations. They also provide further confirmation of the principle discussed earlier: that *all* forms of directional derivatives with respect to the partial velocities reduce to $\partial/\partial u$ and $\partial/\partial v$.

***Shape operator*** The footnote about the reason for the minus sign is questionable, if only because there is always a choice of sign for the unit normal!

**Note:** The definition of shape operator can be extended to act on vector fields (rather than tangent vectors) by the usual pointwise process:

$$S(V):\mathbf{p}\longmapsto S(V(\mathbf{p})).$$

There is a link between the definition of shape operator and the connection forms discussed in *Part II*. There we summarized the connection equations in matrix form as

$$\nabla_V\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix}=\begin{pmatrix}0&\omega_{12}&\omega_{13}\\-\omega_{12}&0&\omega_{23}\\-\omega_{13}&-\omega_{23}&0\end{pmatrix}(V)\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix}$$

for any frame field $E_1$, $E_2$ and $E_3$.

Writing out the last line in full gives

$$\nabla_V E_3=-\omega_{13}E_1-\omega_{23}E_2.$$

Now suppose that we have a frame field defined on a surface such that $E_1$ and $E_2$ are tangent fields and $U=E_3$ is normal. It follows that the shape operator derived from $U$ is given by

$$\begin{aligned}S(V)&=-\nabla_V U\\ &=\omega_{13}(V)E_1+\omega_{23}(V)E_2.\end{aligned}$$

This last equation seems a *good* reason for the introduction of the minus sign into the definition of $S$!

Using the connection forms is clumsier than working with the partial velocities *for surfaces in* $\mathbf{E}^3$. However, when we consider surfaces independent of $\mathbf{E}^3$, we have to abandon unit normals because they belong to the surrounding space, not to the surface. Then we shall have to make use of connection forms for tangent frame fields.

***Example 1.3*** In the exercises we ask you to verify, by direct calculation, the results discussed in the first and third of these examples. Here we consider the plane example.

Suppose that we have a parametrization

$$\mathbf{x} : (u, v) \longmapsto \mathbf{x}(u, v)$$

of a plane $P$ in $\mathbf{E}^3$. Then, since the unit normal vector field $U$ points in a constant direction, the coordinate functions of $U$ must be independent of $u$ and $v$. Hence

$$\begin{aligned} S(\mathbf{x}_u) &= -\nabla_{\mathbf{x}_u} U \\ &= -\frac{\partial U(\mathbf{x}(u,v))}{\partial u} = 0; \\ S(\mathbf{x}_v) &= -\nabla_{\mathbf{x}_v} U \\ &= -\frac{\partial U(\mathbf{x}(u,v))}{\partial v} = 0. \end{aligned}$$

Thus $S$ is the zero operator at all points of $P$.

**Remark**: The appearance of the factor $1/r$ in the shape operator for the sphere should remind you of the appearance of $1/r$ as the curvature of a circle of radius $r$.

***Symmetric operators*** You have probably met $2 \times 2$ symmetric matrices before. Their main property of interest then was that non-singular real, symmetric matrices have real eigenvalues and can always be diagonalized by an orthogonal matrix (actually by a rotation).

*M101*, Block IV.

The same is true of any matrix representing a non-singular, linear transformation which has the generalized symmetric property defined in Lemma 1.4.

***Determinant and trace*** We shall use the trace and determinant of $2 \times 2$ matrices in Section 3. Here we want to mention some properties.

We shall be concerned with the matrices representing shape operators with respect to various bases. Since $S$ is a linear transformation from a two-dimensional space to itself, all such matrices are $2 \times 2$. If $A$ and $B$ are the matrices with respect to two different bases, then they are related by

$$B = P^{-1}AP,$$

where $P$ is non-singular.

It follows that

$$\begin{aligned} \det(B) &= \det(P^{-1}AP) \\ &= \det(P^{-1})\det(A)\det(P) \\ &= (\det(P))^{-1}\det(A)\det(P) \\ &= \det(A). \end{aligned}$$

Thus we may speak of the determinant of the shape operator, rather than of a particular matrix representing it.

The trace of a matrix is the sum of the elements on the leading diagonal and is denoted by trace($A$).

What is not at all obvious is that, with the above notation,

$$\operatorname{trace}(B) = \operatorname{trace}(A).$$

(You can verify this by straightforward, but tedious, algebra.)

Thus determinant and trace are notions belonging to the shape operator, not to its representing matrices.

***The matrix of a shape operator*** From time to time we shall want to write down the the matrix of a shape operator with respect to a particular basis of tangent vectors, usually the partial velocities.

So that there is no danger of writing down the transpose of the correct matrix, we discuss the details here.

Suppose that we have a basis of tangent vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ at a point and we have calculated the effect of $S$ on them and expressed the results in the form

$$\begin{aligned} S(\mathbf{e}_1) &= a\mathbf{e}_1 + b\mathbf{e}_2, \\ S(\mathbf{e}_2) &= c\mathbf{e}_1 + d\mathbf{e}_2. \end{aligned}$$

Suppose that we have a tangent vector

$$\mathbf{v} = p\mathbf{e}_1 + q\mathbf{e}_2$$

at the same point. By the linearity of $S$, we have

$$\begin{aligned} S(\mathbf{v}) &= pS(\mathbf{e}_1) + qS(\mathbf{e}_2) \\ &= p(a\mathbf{e}_1 + b\mathbf{e}_2) + q(c\mathbf{e}_1 + d\mathbf{e}_2) \\ &= (pa + qc)\mathbf{e}_1 + (pb + qd)\mathbf{e}_2. \end{aligned}$$

Expressing the effect of $S$ in terms of the coordinates with respect to $\mathbf{e}_1$ and $\mathbf{e}_2$, we have

$$\begin{pmatrix} p \\ q \end{pmatrix} \longmapsto \begin{pmatrix} pa + qc \\ pb + qd \end{pmatrix},$$

which can be achieved by the matrix

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Note that the coefficients from the images of the basis form the *columns* of the matrix of $S$.

This is actually what you have seen before when constructing matrix transformations from the knowledge of the images of the basis

$$(1, 0) \quad \text{and} \quad (0, 1).$$

There is an important result that *O'Neill* leaves as Exercise 1 in this section. We shall want to make use of it, so we prove it here. You may wish to try the exercise before reading on.

***Lemma 1.5***

Let $M \subset \mathbf{E}^3$ be a surface and let $\alpha$ be a curve in $M$. If $U$ is the restriction of a unit normal vector field on $M$ to $\alpha$, then

$$S(\alpha') = -U',$$

where $S$ is the shape operator derived from $U$.

*Proof* We apply Definition 1.0 from earlier in this commentary. The details are a little messy if we are going to be precise.

Let $\alpha'(t_0)$ be the velocity at $\alpha(t_0)$ on $\alpha$. Then the curve $\beta$, defined by

$$\beta(s) = \alpha(t_0 + s)$$

satisfies

$$\beta(0) = \alpha(t_0) \quad \text{and} \quad \beta'(0) = \alpha'(t_0).$$

By Definition 1.0, we can deduce that

$$\nabla_{\beta'(0)} U = (U(\beta(s)))'(0).$$

But $\beta'(0) = \alpha'(t_0)$ and

$$(U(\beta(s)))'(0) = (U(\alpha(t)))'(t_0),$$

so

$$\begin{aligned} S(\alpha')(t_0) &= -\nabla_{\beta'(0)}U \\ &= -(U(\alpha(t)))'(t_0). \end{aligned}$$

Since $(U(\alpha(t)))'$ is, by definition, $U'$ and since the above calculation holds for all values of $t_0$, we can deduce that

$$S(\alpha') = -U'.$$

The complication in the proof is the need to introduce $\beta$ in order to apply Definition 1.0 directly.

The following exercises are concerned with verifying some of the statements in Example 1.3. The saddle surface can be dealt with by similar means.

**Exercise 1.1** Let $M$ be the cylinder parametrized by

$$\mathbf{x}(u, v) = (r\cos u, r\sin u, v).$$

(a) Use the partial velocities to define an outward unit normal vector field, $U$ on $M$.

(b) Let $S$ be the shape operator on $M$ derived from $U$. Find $S(\mathbf{x}_u(u, v))$ and $S(\mathbf{x}_v(u, v))$.

(c) Find the matrix representing $S$ with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$.

**Exercise 1.2** Let $\Sigma$ be the sphere of radius $r$ parametrized by

$$\mathbf{x}(u, v) = (r\cos u\cos v, r\cos u\sin v, r\sin u).$$

By following a scheme similar to that for the cylinder in the previous exercise, verify the assertion in the text that

$$S(\mathbf{v}) = -\frac{\mathbf{v}}{r}$$

on $\Sigma$.

[*Solutions on page 25* ]

# 2 Normal curvature

---

**Read** *O'Neill, Chapter V, Section 2, pages 195–202.*

---

***Erratum*** *O'Neill*, page 201, in Fig. 5.17, there is a bracket missing. The label for the point on the surface should read:

$$(x, y, f(x, y)). \qquad \blacksquare$$

The main idea introduced in this section is *normal curvature.* This is the formalization of the notion discussed earlier that travelling in different directions on a surface causes you to experience different curvatures.

***Lemma 2.1*** This is an application of one manifestation of the Leibniz property. It will be of great practical importance for the computational techniques in Section 4. Note that heavy use is made of the result

$$S(\alpha') = -U'$$

that we proved at the end of the last section.

***Definition 2.2*** Note that the definition of $k$ restricts its domain to *unit* tangent vectors to $M$.

The remarks following the definition link normal curvature to the (Frenet) curvature of curves in $M$.

The normal curvature function is somewhat unusual, for differential geometry, in that it is *not* linear. It cannot be, since it is defined only on unit vectors.

However, suppose that $\mathbf{v}_{\mathrm{p}}$ is non-zero and tangent to $M$. Then

$$\mathbf{u} = \frac{\mathbf{v}}{||\mathbf{v}||}$$

is a unit vector and

$$\begin{aligned} k(\mathbf{u}) &= S(\mathbf{u}) \cdot \mathbf{u} \\ &= S\left(\frac{\mathbf{v}}{||\mathbf{v}||}\right) \cdot \left(\frac{\mathbf{v}}{||\mathbf{v}||}\right) \\ &= \frac{1}{||\mathbf{v}||^2} S(\mathbf{v}) \cdot (\mathbf{v}), \end{aligned}$$

by the linearity of $S$ and dot products.

Thus

$$S(\mathbf{v}) \cdot \mathbf{v} = ||\mathbf{v}||^2 k(\mathbf{u}),$$

where $\mathbf{u}$ is a unit vector in the direction of $\mathbf{v}$. This last equation can be used to find the normal curvature in the direction of a given non-unit tangent vector (often a partial velocity).

***Example*** We look algebraically at the saddle surface example in the last paragraph of page 198.

The surface can be covered by the single Monge patch

$$\mathbf{x}(u, v) = (u, v, uv).$$

The partial velocities are

$$\mathbf{x}_u(u, v) = (1, 0, v) \quad \text{and} \quad \mathbf{x}_v(u, v) = (0, 1, u).$$

We can construct a unit normal vector field from the partial velocities. This process gives

$$\begin{aligned} U(\mathbf{x}(u, v)) &= \frac{\mathbf{x}_u \times \mathbf{x}_v}{||\mathbf{x}_u \times \mathbf{x}_v||} \\ &= \frac{(-v, -u, 1)}{\sqrt{1 + u^2 + v^2}}. \end{aligned}$$

In order to find normal curvatures we need to know about the shape operator. As usual, we first find its effect on the partial velocities by partial differentiation. The details are a little messy, but careful differentiation yields

$$\begin{aligned} \frac{\partial U}{\partial u} &= \frac{(uv, -(1 + v^2), -u)}{(1 + u^2 + v^2)^{3/2}} \\ \frac{\partial U}{\partial v} &= \frac{(-(1 + u^2), uv, -v)}{(1 + u^2 + v^2)^{3/2}}. \end{aligned}$$

It follows that

$$\begin{aligned} S(\mathbf{x}_u) &= -\frac{\partial U}{\partial u} \\ &= \frac{(-uv, 1+v^2, u)}{(1+u^2+v^2)^{3/2}}, \end{aligned}$$

$$\begin{aligned} S(\mathbf{x}_v) &= -\frac{\partial U}{\partial v} \\ &= \frac{(1+u^2, -uv, v)}{(1+u^2+v^2)^{3/2}}. \end{aligned}$$

In the special case discussed in *O'Neill*, we have $u = v = 0$. Here

$$\begin{aligned} \mathbf{x}_u(0,0) &= (1,0,0), \\ \mathbf{x}_v(0,0) &= (0,1,0), \\ S(\mathbf{x}_u) &= (0,1,0), \\ S(\mathbf{x}_v) &= (1,0,0). \end{aligned}$$

Because the partial velocities at the origin are unit length, we have

$$\begin{aligned} k(\mathbf{x}_u) &= S(\mathbf{x}_u) \cdot \mathbf{x}_u \\ &= (0,1,0) \cdot (1,0,0) \\ &= 0, \end{aligned}$$

$$\begin{aligned} k(\mathbf{x}_v) &= (1,0,0) \cdot (0,1,0) \\ &= 0. \end{aligned}$$

These agree with *O'Neill*.

Now consider the direction of the line $x = y$ in the plane $z = 0$. One vector in this direction is

$$\mathbf{v} = \mathbf{x}_u(0,0) + \mathbf{x}_v(0,0) = (1,1,0)$$

and a unit vector in the same direction is

$$\mathbf{u} = \frac{1}{\sqrt{2}}(1,1,0).$$

By linearity,

$$S(\mathbf{u}) = \frac{1}{\sqrt{2}}(S(\mathbf{x}_u) + S(\mathbf{x}_v)) = \frac{1}{\sqrt{2}}(1,1,0).$$

Thus

$$\begin{aligned} k(\mathbf{u}) &= \frac{1}{\sqrt{2}}(1,1,0) \cdot \frac{1}{\sqrt{2}}(1,1,0) \\ &= 1. \end{aligned}$$

We can look at the direction of $y = -x$ by considering $\mathbf{x}_u - \mathbf{x}_v$. A suitable unit vector is

$$\mathbf{w} = \frac{1}{\sqrt{2}}(1,-1,0),$$

leading to

$$k(\mathbf{w}) = -1.$$

***Principal curvatures and directions*** In spite of the assertion about being able to 'pick out' principal directions, one of the course's aims is to have computational tools for finding principal curvatures and directions.

The proof of Theorem 2.5 is a first-principles proof of the eigenvalue and eigenvector properties of symmetric linear transformations on two-dimensional spaces.

Where we refer to eigenvalues and eigenvectors, *O'Neill* uses the terms *characteristic* values and *characteristic* vectors.

We can summarize the results as follows. At each point of the surface:

(a) there is an orthonormal basis of eigenvectors of $S$;

(b) the eigenvalues of $S$ are the extreme values of the normal curvature at that point, that is, they are the principal curvatures at that point;

(c) the eigenvectors define the principal directions at that point.

These results tie the shape of the surface to the algebraic properties of the shape operator $S$.

It is sometimes possible to find the matrix representing $S$ with respect to the partial velocities fairly easily. From this matrix, we can calculate the eigenvalues and eigenvectors of $S$ quite simply.

Sometimes the calculation of the matrix is not straightforward.

We pursue the saddle surface example a little further. From our calculations earlier, we can deduce (after some working) that

$$S(\mathbf{x}_u) = \frac{1}{(1+u^2+v^2)^{3/2}}(-uv\mathbf{x}_u + (1+v^2)\mathbf{x}_v),$$

$$S(\mathbf{x}_v) = \frac{1}{(1+u^2+v^2)^{3/2}}((1+u^2)\mathbf{x}_u - uv\mathbf{x}_v).$$

The matrix of $S$ with respect to the partial velocities is, therefore,

$$\begin{pmatrix} \dfrac{-uv}{(1+u^2+v^2)^{3/2}} & \dfrac{1+u^2}{(1+u^2+v^2)^{3/2}} \\[2ex] \dfrac{1+v^2}{(1+u^2+v^2)^{3/2}} & \dfrac{-uv}{(1+u^2+v^2)^{3/2}} \end{pmatrix}.$$

Here there is scope for writing down the transpose of the correct matrix. Working with the transpose would make no difference to the determinant, trace or eigenvalues but the eigenvector calculations would be incorrect

The general expressions for the eigenvalues of this matrix are far from simple! However, at $u = v = 0$, the matrix reduces to

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

which has eigenvalues 1 and $-1$. The usual method of finding eigenvectors yields $(1,1)$ and $(1,-1)$ as the coordinates with respect to the partial velocities. Thus the principal directions at the origin are in the directions of

$$\mathbf{x}_u(0,0) \pm \mathbf{x}_v(0,0),$$

which agrees with those we obtained earlier.

***Umbilic points*** Definition 2.4 and Theorem 2.5 can be combined to give a practical test for umbilic points.

The function

$$k(\mathbf{u}) = k_1 \cos^2\vartheta + k_2 \sin^2\vartheta,$$

of Corollary 2.6, is a continuous function of $\vartheta$. Because the eigenvalues $k_1$ and $k_2$ of the shape operator are the maximum and minimum values of a continuous function, we can deduce that $k(\mathbf{u})$ is constant if, and only if,

$$k_1 = k_2.$$

Now, if we have a matrix for $S$, we can express the condition for the eigenvalue equation to have equal roots. Suppose that the matrix is

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Then the eigenvalue equation reduces to

$$k^2 - (a+d)k + (ad - bc) = 0.$$

It is rather more revealing to write this in the form

$$k^2 - \operatorname{trace}(A)\,k + \det(A) = 0.$$

The quadratic equation formula gives '$b^2 = 4ac$' as the condition for equal roots. In the case of the above eigenvalue equation, this becomes

$$(\text{trace}(A))^2 = 4 \times \det(A).$$

Let us apply this test to the saddle surface example. From the matrix

$$A = \begin{pmatrix} \dfrac{-uv}{(1+u^2+v^2)^{3/2}} & \dfrac{1+u^2}{(1+u^2+v^2)^{3/2}} \\[2ex] \dfrac{1+v^2}{(1+u^2+v^2)^{3/2}} & \dfrac{-uv}{(1+u^2+v^2)^{3/2}} \end{pmatrix},$$

we have

$$\text{trace}(A) = \frac{-2uv}{(1+u^2+v^2)^{3/2}},$$

$$\det(A) = \frac{u^2v^2 - (1+u^2)(1+v^2)}{(1+u^2+v^2)^3}.$$

Thus, a point with parameters $(u, v)$ will be umbilic if, and only if,

$$\left(\frac{-2uv}{(1+u^2+v^2)^{3/2}}\right)^2 = 4\left(\frac{u^2v^2 - (1+u^2)(1+v^2)}{(1+u^2+v^2)^3}\right).$$

Both sides have the same denominator so the condition reduces to

$$u^2v^2 = u^2v^2 - (1+u^2)(1+v^2).$$

Since the second term on the right-hand side must be at least 1, there can be no solutions. We deduce that the saddle surface has no umbilic points.

**Exercise 2.1** This question concerns the cylinder $M$ parametrized by the mapping

$$\mathbf{x}(u, v) = (r\cos u, r\sin u, v), \qquad r > 0.$$

It is a continuation of Exercise 1.1 using the notation and results of that exercise.

(a) Find the eigenvalues and eigenvectors of the shape operator $S$ and, hence, the principal curvatures and directions.

(b) Does $M$ have any umbilic points?

(c) Find the normal curvature in the direction of $\mathbf{x}_u + \mathbf{x}_v$.

**Exercise 2.2** *O'Neill*, page 202, Exercise 3. Note that the curves given can be defined by $\alpha(t) = \mathbf{x}(t, \pm t^n)$, where $\mathbf{x}$ is the parametrization in Exercise 2.1.

[*Solutions on page 25* ]

# 3 Gaussian curvature

---

**Read** *O'Neill, Chapter V, Section 3, pages 203–207.*

---

**Erratum** *O'Neill*, page 204, ninth line and last but one line; page 205, four lines from the bottom of the page (first line of the proof) there are occurrences where:

for $T_{\mathbf{p}}(M)$ read $\overline{T}_{\mathbf{p}}(M)$. ■

This section introduces two new definitions, discusses some examples and proves the results that will be the basis of some straightforward computational techniques.

**Gaussian and mean curvature** We remarked earlier that the determinant and trace were the same for all matrices representing a shape operator. Hence Gaussian and mean curvatures are well defined.

Note that both $K$ and $H$ are functions from the surface to $\mathbf{R}$.

In the examples *O'Neill* indicates how the sign of $K$ can be deduced by knowing the shape (in the intuitive sense) of the surface. Once the link between the formal definitions and intuition is established, we usually calculate the $K$ and $H$ functions to *provide* information about a new surface.

**Examples** As *O'Neill* says, we shall check a number of the results in these examples once we have efficient methods of calculation.

In *theory* what we have already would enable us to calculate $K$ and $H$ for, say, the monkey saddle parametrized by

$$\begin{aligned}\mathbf{x}(u,v) &= (u, v, u(u+\sqrt{3}v)(u-\sqrt{3}v)) \\ &= (u, v, u(u^2-3v^2)).\end{aligned}$$

In practice, we run into the same difficulties as we did when trying to apply directly the Frenet formulas for non-unit speed curves: some of the differentiation becomes so involved as to invite errors.

To illustrate this point, consider the monkey saddle parametrization above. We can find the partial velocities:

$$\begin{aligned}\mathbf{x}_u &= (1, 0, 3u^2-3v^2) \\ &= (1, 0, 3(u^2-v^2)), \\ \mathbf{x}_v &= (0, 1, -6uv).\end{aligned}$$

If we construct a unit normal vector field from these, we have

$$\begin{aligned}\mathbf{x}_u \times \mathbf{x}_v &= (-3(u^2-v^2), 6uv, 1), \\ \|\mathbf{x}_u \times \mathbf{x}_v\|^2 &= 9(u^2-v^2)^2 + 36u^2v^2 + 1 \\ &= 9(u^2+v^2)^2 + 1\end{aligned}$$

and so

$$U(\mathbf{x}(u,v)) = \frac{(-3(u^2-v^2), 6uv, 1)}{\sqrt{9(u^2+v^2)^2+1}}.$$

Now, finding $S(\mathbf{x}_u)$ etc. requires the partial derivatives of $U$. These will be involved expressions and it is unlikely to be easy to find the matrix representation of $S$ with respect to $\mathbf{x}_u$ and $\mathbf{x}_v$.

The way out of such problems is similar to that for curves. We shall obtain formulas using higher derivatives of the parametrization.

**Lemma 3.4** This result and the remarks following it provide the foundation for easier calculations. We shall apply it to the usual special case, where $\mathbf{v}$ and $\mathbf{w}$ are the partial velocities.

As Lemma 3.4 stands, it is no more use than the attempt above, since $S(\mathbf{v})$ etc. still appear, requiring partial differentiation of $U$. However, as so often, the Leibniz property will enable us to simplify things.

This simplification is in the next section.

The formulas after the end of the proof of Lemma 3.4 will appear in more memorable form later on.

**Gaussian curvature and connection forms** In Section 1, we discussed the link between the shape operator and the connection forms of a frame field made up of two unit tangent vector fields $E_1$ and $E_2$, together with a unit normal vector field

$$U = E_1 \times E_2.$$

We obtained

$$\begin{aligned}S(V) &= -\nabla_V U\\ &= \omega_{13}(V)E_1 + \omega_{23}(V)E_2,\end{aligned}$$

for any vector field $V$ on the surface.

If we apply this to $E_1$ and $E_2$, we obtain

$$\begin{aligned}S(E_1) &= \omega_{13}(E_1)E_1 + \omega_{23}(E_1)E_2,\\ S(E_2) &= \omega_{13}(E_2)E_1 + \omega_{23}(E_2)E_2.\end{aligned}$$

The matrix of $S$ with respect to $E_1, E_2$ is, therefore,

$$\begin{pmatrix}\omega_{13}(E_1) & \omega_{13}(E_2)\\ \omega_{23}(E_1) & \omega_{23}(E_2)\end{pmatrix},$$

with determinant

$$\omega_{13}(E_1)\omega_{23}(E_2) - \omega_{13}(E_2)\omega_{23}(E_1) = (\omega_{13}\wedge\omega_{23})(E_1, E_2).$$

We can obtain a further link by considering the second structural equations. In matrix form they become

*Part II*, Section 8.

$$d\omega = \omega \wedge \omega.$$

If we write out the $d\omega_{12}$ entry in full, we have

This is the second entry of the first row of the product.

$$\begin{aligned}d\omega_{12} &= \omega_{11}\wedge\omega_{12} + \omega_{12}\wedge\omega_{22} + \omega_{13}\wedge\omega_{32}\\ &= 0\wedge\omega_{12} + \omega_{12}\wedge 0 - \omega_{13}\wedge\omega_{23}\\ &= -\omega_{13}\wedge\omega_{23}.\end{aligned}$$

Combining this with our earlier calculations, we see that

$$\begin{aligned}d\omega_{12}(E_1, E_2) &= -(\omega_{13}\wedge\omega_{23})(E_1, E_2)\\ &= -K.\end{aligned}$$

Thus

$$K = -d\omega_{12}(E_1, E_2).$$

This last equation should be interpreted as

$$K(\mathbf{p}) = -d\omega_{12}(E_1(\mathbf{p}), E_2(\mathbf{p})),$$

for each point $\mathbf{p}$ on the surface.

Thus, if we can find independent means of calculating connection forms on a surface, then we can calculate Gaussian curvature.

We shall pursue this line in *Part VI.*

**Exercise 3.1** *O'Neill*, page 207, Exercise 1.

**Exercise 3.2** Let $M$ be the saddle surface parametrized by

$$\mathbf{x}(u, v) = (u, v, uv).$$

Use the matrix of the shape operator found in Section 2 to find the Gaussian and mean curvatures.

Does $M$ have any planar points?

**Exercise 3.3** *O'Neill*, page 207, Exercise 3. (*Hint*: The notation of Section 2, Corollary 2.6 should prove useful.)

[*Solutions on page 26* ]

# 4 Computational techniques

---

**Read** *O'Neill, Chapter V, Section 4, pages 210–219.*

---

### Errata

1 *O'Neill*, page 210, in the third displayed line, the expression for $\mathbf{w}$ should read:

$$\mathbf{w} = w_1\mathbf{x}_u + w_2\mathbf{x}_v.$$

2 *O'Neill*, page 214, in the final display line, the inequalities for $K$ should read:

$$-\frac{1}{b^2} \leq K < 0.$$

3 *O'Neill*, page 218, in the last paragraph and in the diagram, the two occurrences of $T_{\mathbf{p}}(M)$ should read:

$$\overline{T}_{\mathbf{p}}(M). \qquad \blacksquare$$

The main aim of this section is to provide computationally efficient formulas for Gaussian and mean curvature. We shall use the partial velocities to build up such formulas.

**Precision and convenience** Near the beginning of the reading passage *O'Neill* adopts a rather more precise approach to exactly what the partial velocities are. It is worth looking hard at what he is saying and comparing it with what it is convenient to assume.

Let us look at a simple specific example to illustrate what the fuss is about. Consider the saddle surface $M$ parametrized by the single proper patch $\mathbf{x}$ defined by

$$\mathbf{x}(u,v) = (u, v, uv), \quad (u,v) \in \mathbf{E}^2.$$

As we have observed much earlier, the partial velocities

$$\mathbf{x}_u(u,v) = (1, 0, v)$$
$$\mathbf{x}_v(u,v) = (0, 1, u)$$

are tangent vectors to $M$ at the point $\mathbf{x}(u,v)$ and are the velocity vectors of the $u$-parameter and $v$-parameter curve passing through that point.

Thus, $\mathbf{x}_u$ and $\mathbf{x}_v$ are functions *from the domain of* $\mathbf{x}$ to the collection of all tangent vectors to $M$. The partial velocities are not, strictly, vector fields on $M$. Such vector fields have domain $M$, not $\mathbf{E}^2$.

That said, however, there are two vector fields that we can define on $M$:

$$\mathbf{x}(u,v) \longmapsto \mathbf{x}_u(u,v),$$
$$\mathbf{x}(u,v) \longmapsto \mathbf{x}_v(u,v).$$

There seems to be little harm in referring to these by the same names as the partial velocities, so long as we realize that it *is* an abuse of notation. (It is similar to the abuse involved in not distinguishing between $f$ and $f(\mathbf{x})$ for a function on a surface.)

The discussion above generalizes from the saddle surface to any surface covered by a single patch. It also applies whenever there is a parametrization onto the whole surface. The case where it cannot automatically be applied is where the surface is defined by several overlapping patches. The difficulty is that the partial velocities of different, overlapping patches may well not agree on the overlap.

However, as discussed earlier, the unit normal defined as

*Part IV*, Section 6.

$$U = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$$

*is* well defined, even where patches overlap. There is a small abuse of notation in referring to $U$ as a unit normal vector field on the surface, instead of a unit normal vector valued function on the domain of $\mathbf{x}$. The abuse amounts to referring to both

$$(u,v) \longmapsto \frac{\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)}{\|\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)\|}$$

and

$$\mathbf{x}(u,v) \longmapsto \frac{\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)}{\|\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)\|}$$

as $U$. Having mentioned that this *is* abuse, it seems fairly harmless to us!

***Higher derivatives*** The notation $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$ and $\mathbf{x}_{vv}$ for the second derivatives is consistent with that for partial velocities.

***The functions $l$, $m$ and $n$*** The motivation for introducing these functions stems from the expressions for $K$ and $H$ (*O'Neill*, page 206). In those formulas, expressions such as

$$S(V)\cdot V,\ S(V)\cdot W \quad \text{and} \quad S(W)\cdot V$$

appear.

If we apply the formula for $K$ (page 206) to the special case, where $V$ is the vector field defined by $\mathbf{x}_u$ and $W$ corresponds to $\mathbf{x}_v$, then we obtain

$$\begin{aligned} K &= \frac{\begin{vmatrix} S(\mathbf{x}_u)\cdot\mathbf{x}_u & S(\mathbf{x}_u)\cdot\mathbf{x}_v \\ S(\mathbf{x}_v)\cdot\mathbf{x}_u & S(\mathbf{x}_v)\cdot\mathbf{x}_v \end{vmatrix}}{\begin{vmatrix} \mathbf{x}_u\cdot\mathbf{x}_u & \mathbf{x}_u\cdot\mathbf{x}_v \\ \mathbf{x}_v\cdot\mathbf{x}_u & \mathbf{x}_v\cdot\mathbf{x}_v \end{vmatrix}} \\ &= \frac{\begin{vmatrix} l & m \\ m & n \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}} \\ &= \frac{ln - m^2}{EG - F^2}. \end{aligned}$$

Similar considerations give a formula for $H$.

These three new functions, together with $E$, $F$ and $G$ also appear naturally in a simplistic approach to finding matrices of shape operators, which we now indicate.

Suppose that $S$ is the shape operator derived from a unit normal vector field $U$. Then, at any point on the surface, we have

$$\begin{aligned} S(\mathbf{x}_u) &= a\mathbf{x}_u + b\mathbf{x}_v, \\ S(\mathbf{x}_v) &= c\mathbf{x}_u + d\mathbf{x}_v, \end{aligned}$$

for suitable coefficients $a, b, c, d$. We can obtain enough simultaneous linear equations to find the coefficients if we take the dot product of both these equations with both partial velocities.

This gives the following equations.

$$\begin{aligned} S(\mathbf{x}_u)\cdot\mathbf{x}_u &= a\mathbf{x}_u\cdot\mathbf{x}_u + b\mathbf{x}_v\cdot\mathbf{x}_u, \\ S(\mathbf{x}_u)\cdot\mathbf{x}_v &= a\mathbf{x}_u\cdot\mathbf{x}_v + b\mathbf{x}_v\cdot\mathbf{x}_v, \\ S(\mathbf{x}_v)\cdot\mathbf{x}_u &= c\mathbf{x}_u\cdot\mathbf{x}_u + d\mathbf{x}_v\cdot\mathbf{x}_u, \\ S(\mathbf{x}_v)\cdot\mathbf{x}_v &= c\mathbf{x}_u\cdot\mathbf{x}_v + d\mathbf{x}_v\cdot\mathbf{x}_v. \end{aligned}$$

Using the definitions of the various functions, these equations become

$$\begin{aligned} l &= aE + bF, \\ m &= aF + bG, \\ m &= cE + dF, \\ n &= cF + dG. \end{aligned}$$

The usual methods for solving linear simultaneous equations give

$$a = \frac{Gl - Fm}{EG - F^2},$$
$$b = \frac{-Fl + Gm}{EG - F^2},$$
$$c = \frac{Gm - Fn}{EG - F^2},$$
$$d = \frac{-Fm + Gn}{EG - F^2}.$$

This gives the following, rather unwieldy, matrix for $S$.

$$\begin{pmatrix} \dfrac{Gl - Fm}{EG - F^2} & \dfrac{Gm - Fn}{EG - F^2} \\[2ex] \dfrac{-Fl + Gm}{EG - F^2} & \dfrac{-Fm + Gn}{EG - F^2} \end{pmatrix}.$$

Although the matrix is unwieldy, expressions for its trace and determinant are relatively straightforward.

$$\begin{aligned} \det(S) &= \frac{Gl - Fm}{EG - F^2}\frac{-Fm + Gn}{EG - F^2} - \frac{-Fl + Gm}{EG - F^2}\frac{Gm - Fn}{EG - F^2} \\ &= \frac{(ln - m^2)(EG - F^2)}{(EG - F^2)^2} \\ &= \frac{ln - m^2}{EG - F^2} \quad \text{(after some algebra).} \end{aligned}$$

$$\begin{aligned} \operatorname{trace}(S) &= \frac{Gl - Fm}{EG - F^2} + \frac{-Fm + En}{EG - F^2} \\ &= \frac{Gl + En - 2Fm}{EG - F^2}. \end{aligned}$$

This 'first principles' approach yields the same formulas as Corollary 4.1.

***Lemma 4.2*** This is the key to making the formulas of Corollary 4.1 usable by avoiding the need to calculate partial derivatives of $U$.

The proof of the lemma uses three applications of the Leibniz property.

$$0 = \frac{\partial}{\partial u}(U \cdot \mathbf{x}_u) = U_u \cdot \mathbf{x}_u + U \cdot \mathbf{x}_{uu}$$
$$0 = \frac{\partial}{\partial u}(U \cdot \mathbf{x}_v) = U_u \cdot \mathbf{x}_v + U \cdot \mathbf{x}_{uv}$$
$$0 = \frac{\partial}{\partial v}(U \cdot \mathbf{x}_v) = U_v \cdot \mathbf{x}_v + U \cdot \mathbf{x}_{vv}$$

These are then interpreted by using $U_u = -S(\mathbf{x}_u)$ etc. to give

$$l = U \cdot \mathbf{x}_{uu}$$
$$m = U \cdot \mathbf{x}_{uv}$$
$$n = U \cdot \mathbf{x}_{vv}.$$

***Example*** We shall now use these formulas and Lemma 4.2 to pursue the saddle surface example. The following provides a little more detail than is given in *O'Neill*.

We have already obtained the following.

$$\mathbf{x}_u = (1, 0, v),$$
$$\mathbf{x}_v = (0, 1, u),$$
$$U = \frac{(-v, -u, 1)}{\sqrt{1 + u^2 + v^2}}.$$

It follows that

$$E = \mathbf{x}_u \cdot \mathbf{x}_u = 1 + v^2,$$

$$F = \mathbf{x}_u \cdot \mathbf{x}_v = uv,$$

$$G = \mathbf{x}_v \cdot \mathbf{x}_v = 1 + u^2;$$

$$EG - F^2 = (1+v^2)(1+u^2) - u^2v^2 = 1 + u^2 + v^2.$$

The second derivatives are

$$\mathbf{x}_{uu} = (0,0,0),$$
$$\mathbf{x}_{uv} = (0,0,1),$$
$$\mathbf{x}_{vv} = (0,0,0).$$

Hence

$$l = U \cdot \mathbf{x}_{uu} = 0,$$

$$m = U \cdot \mathbf{x}_{uv} = \frac{1}{\sqrt{1+u^2+v^2}},$$

$$n = U \cdot \mathbf{x}_{vv} = 0.$$

Applying the formulas, we have

$$K = \frac{ln - m^2}{EG - F^2} = \frac{0 - (1/(\sqrt{1+u^2+v^2}))^2}{1+u^2+v^2} = \frac{-1}{(1+u^2+v^2)^2};$$

$$H = \frac{Gl + En - 2Fm}{2(EG - F^2)} = \frac{0 + 0 - 2(uv/\sqrt{1+u^2+v^2})}{2(1+u^2+v^2)} = \frac{-uv}{(1+u^2+v^2)^{3/2}}.$$

These are the same as obtained previously by different methods.

This example illustrates the benefit of the new methods. Defining a unit normal vector field is still required but partial differentiation of this unit normal is replaced by differentiating the partial velocities, often a much easier task.

Should the principal curvatures be required, they can be found as solutions of Corollary 3.5

$$k^2 - 2Hk + K = 0.$$

***Lemma 4.4*** We shall not use this result as much as the previous ones but, for completeness, we indicate the 'routine' deduction from Lemma 3.4. We suspect that *O'Neill* omitted the proof because of the algebra involved!

Firstly, we need a unit normal vector field. This is provided by reducing $Z$ to unit length:

$$U = \frac{Z}{\|Z\|}.$$

Now we express $S(V)$ and $S(W)$ in terms of $Z$.

$$\begin{aligned} S(V) &= -\nabla_V U \\ &= -\nabla_V \frac{Z}{\|Z\|} \\ &= -\left(\frac{1}{\|Z\|}\nabla_V Z + V\left[\frac{1}{\|Z\|}\right] Z\right) \end{aligned}$$

(by the Leibniz formula for covariant derivatives).

Similarly,

$$S(W) = -\left(\frac{1}{\|Z\|}\nabla_W Z + W\left[\frac{1}{\|Z\|}\right] Z\right).$$

Taking the cross product and omitting zeros, we have

$$S(V) \times S(W) = \frac{1}{\|Z\|^2}(\nabla_V Z \times \nabla_W Z) + \frac{1}{\|Z\|}\left(V\left[\frac{1}{\|Z\|^2}\right]\nabla_V Z \times Z + V\left[\frac{1}{\|Z\|^2}\right] Z \times \nabla_W Z\right).$$

When we take the dot product with $Z$, the last two terms disappear, so

$$Z \cdot (S(V) \times S(W)) = \frac{1}{\|Z\|^2} Z \cdot (\nabla_V Z \times \nabla_W Z).$$

From Lemma 3.4, we have

$$S(V) \times S(W) = KV \times W$$

so, taking the dot product with $Z$, we have

$$\begin{aligned} Z \cdot (S(V) \times S(W)) &= KZ \cdot V \times W \\ &= KZ \cdot Z \quad \text{(definition of } Z \text{ etc.)} \\ &= K\|Z\|^2. \end{aligned}$$

Combining with the previous result, we have

$$\frac{1}{\|Z\|^2} Z \cdot (\nabla_V Z \times \nabla_W Z) = K\|Z\|^2,$$

from which the first result follows.

The calculations for $H$ are much the same and we omit them.

***Example 4.5*** This shows that applying Lemma 4.4 can require some ingenuity in the choice of $V$ and $W$. We shall not make a great deal of use of this technique.

***Patches and parametrizations*** The final remarks of this section of *O'Neill* show that calculations are much easier if we can find a single parametrization of the surface being studied. This will be the case in most of our work.

**Exercise 4.1** *O'Neill*, page 219, Exercise 2. **Note:** Subscripts on $f$ denote partial differentiation.

**Exercise 4.2** *O'Neill*, page 219, Exercise 3.

**Exercise 4.3** *O'Neill*, page 219, Exercise 4.

[*Solutions on page 27*]

# 5 Special curves

---

***Read*** *O'Neill, Chapter V, Section 5, pages 223–229.*

---

### *Errata*

1 *O'Neill*, page 224, the tenth line should read:

'...(unit) principal vectors belonging to ... .'

2 *O'Neill*, page 228, the second line should read:

'...$S(\alpha') = -U'$ ... .' ■

Principal and asymptotic curves.

There is a sense in which two of the three types of curve studied here have definitions that follow naturally from what has gone before.

We have singled out the principal directions at a point on a surface as being the directions where the normal curvature is extreme. It is not unreasonable, therefore, to consider curves that always point in a principal direction at each point on their routes.

Equally, directions in which the normal curvature is zero are of some interest since they can occur on surfaces which are most definitely curved in the intuitive sense.

The third type of curve—geodesics—arise from a different consideration: the search for the quickest way to get from one point to another of a surface *staying on the surface.*

Think about travelling around at constant *speed* on a surface. Because of the curvature of the surface some acceleration normal to the surface is more or less inevitable. However, intuitively, any acceleration tangent to the surface represents *avoidable* cornering. Thus, the quickest way to travel from one point to another is by a route whose acceleration is always normal to the surface.

The discussion above is intended as a brief motivation for the three definitions: 5.1, 5.5 and 5.7.

Together with the definitions, *O'Neill* proves results designed to give usable tests to see if a given curve is principal, asymptotic or a geodesic. In practice some of the tests are more usable than others.

***Lemma 5.2*** As a practical test, $U'$ being collinear with $\alpha'$ suffers from the need to calculate $U'$. As we have seen for the saddle surface, this can be sufficiently involved to be error-inducing.

***Asymptotic curves*** The final test in the paragraph following Definition 5.5 is more usable than that contained in Lemma 5.2 because it involves the second derivative of the curve rather than $U'$.

***Geodesics*** The paragraph following the definition can be made to yield a specific test if we use the fact that the partial velocities of a patch (or parametrization) provide a basis for the tangent plane at each point. Therefore $\alpha$ is a geodesic if, and only if,

$$\alpha'' \cdot \mathbf{x}_u = \alpha'' \cdot \mathbf{x}_v = 0,$$

for all points on the route of $\alpha$. This result is essentially stating that a vector is perpendicular to a plane precisely when it is orthogonal to a basis for the plane.

The tests are summarized on page 229, just before the exercises. Where there is more than one test, you only need show that *one* is satisfied, the tests in any one row of the table are equivalent.

***General*** The search for special curves in a surface usually involves the solution of differential equations. To see why, consider a surface $M$ parametrized by $\mathbf{x}$. Suppose that

$$\alpha = \mathbf{x}(\alpha_1, \alpha_2)$$

is a curve in $M$.

The chain rule enables $\alpha'$ and $\alpha''$ to be expressed in terms of

$$\mathbf{x}_u\,,\ \mathbf{x}_v\,,\ \mathbf{x}_{uu}\,,\ \mathbf{x}_{uv}\,,\ \mathbf{x}_{vv}$$

and the derivatives of $\alpha_1$ and $\alpha_2$. Thus all the tests discussed in the section can be written in terms of differential equations satisfied by the coordinate functions $\alpha_1$ and $\alpha_2$.

**Example** As a simple example, let us apply some of these ideas to the information that we have already assembled about the saddle surface $M$ parametrized by

$$\mathbf{x}(u,v) = (u, v, uv).$$

We consider a curve $\alpha = \mathbf{x}(\alpha_1, \alpha_2)$ in $M$ and, rather than use the chain rule, we use the partial velocities and unit normal vector field already derived.

On the curve

$$\begin{aligned} U(\alpha) &= \frac{(-\alpha_2, -\alpha_1, 1)}{\sqrt{1+\alpha_1^2+\alpha_2^2}}, \\ \mathbf{x}_u(\alpha) &= (1, 0, \alpha_2), \\ \mathbf{x}_v(\alpha) &= (0, 1, \alpha_1), \\ \alpha' &= (\alpha_1', \alpha_2', \alpha_1'\alpha_2 + \alpha_1\alpha_2'), \\ \alpha'' &= (\alpha_1'', \alpha_2'', \alpha_1''\alpha_2 + 2\alpha_1'\alpha_2' + \alpha_1\alpha_2''). \end{aligned}$$

We consider the case where $\alpha$ is asymptotic. Using the test

$$U \cdot \alpha'' = 0,$$

we obtain, after some algebra,

$$2\alpha_1'\alpha_2' = 0.$$

The regularity of $\alpha$ prevents both the derivatives being zero together. Hence the solutions are

$$\alpha_1 = \text{constant} \quad \text{or} \quad \alpha_2 = \text{constant}.$$

The asymptotic curves in $M$ are, therefore, described by the condition that exactly one of the parameters must be held constant; that is, they are the parameter curves.

This result does correspond to the results obtained previously. We have shown that

$$l = U \cdot \mathbf{x}_{uu} = 0 \quad \text{and} \quad n = U \cdot \mathbf{x}_{vv} = 0.$$

But $\mathbf{x}_{uu}$ and $\mathbf{x}_{vv}$ are the accelerations of the parameter curves so

$$l = n = 0$$

says precisely that the parameter curves are asymptotic. ■

This example does illustrate the point about the appearance of differential equations. They are not usually as easy to solve!

**Exercise 5.1** *O'Neill*, page 229, Exercise 1.

**Exercise 5.2** *O'Neill*, page 230, Exercise 2. (*Hint*: You may wish to use the parametrization

$$\mathbf{x}(u,v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u), \quad R > r > 0$$

for the torus. You may also wish to use the Monge patch for the saddle surface.)

**Exercise 5.3** Let $M \subset \mathbf{E}^3$ be a surface and let $\alpha$ be a unit-speed asymptotic curve in $M$ with positive curvature $\kappa$. Using the usual notation for the Frenet apparatus of $\alpha$ show that:

(a) $B$ is normal to the surface along $\alpha$;

(b) $S(T) = \tau N$.

[*Solutions on page 27*]

# 6 Surfaces of revolution

**Read** *O'Neill, Chapter V, Section 6, pages 235–242.*

***Erratum*** *O'Neill*, page 241, five lines from the bottom, for the expressions given for $k_\mu$ and $k_\pi$, read:

$$k_\mu = \frac{h'}{c} \quad \text{and} \quad k_\pi = \frac{-1}{ch'}.$$ ■

This section introduces no new ideas but provides some further examples of the use of the techniques discussed so far.

In a sense some of the work is 'general' in that it discusses the class of *all* surfaces of revolution. It is, however, at a lower level of generality than much of the theory.

On the whole *O'Neill* adopts the pragmatic approach to defining tangent vector fields and unit normal vector fields in terms of the partial velocities.

We have discussed the abuse of notation embodied in this approach at several points in the course.

Whilst it is useful to have formulas for the various curvatures of a general surface of revolution, in practice it is often simpler to deal with each case using the standard computational methods of Section 4.

One factor that appears in most of the formulas is the square of the speed of the profile curve:

$$E = g'^2 + h'^2.$$

If it is possible to arrange a unit-speed parametrization of the profile curve, then these formulas are simplified considerably.

***Orthogonal parametrizations*** There is a more general point that arises from the discussion of surfaces of revolution.

A number of simplifications in the calculations arise because the partial velocities are orthogonal, that is,

$$F = \mathbf{x}_u \cdot \mathbf{x}_v = 0.$$

The most obvious simplification is in the term

$$EG - F^2$$

which appears in the formulas for both $K$ and $H$.

A slightly less obvious one is that the partial velocities can be used to construct a *frame* field on the surface by using:

$$E_1 = \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} = \frac{\mathbf{x}_u}{\sqrt{E}},$$

$$E_2 = \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|} = \frac{\mathbf{x}_v}{\sqrt{G}},$$

$$U = E_1 \times E_2.$$

The benefit from this is that, should you want the matrix of the shape operator with respect to this basis, the coefficients can be obtained by applying orthonormal expansion to $S(E_1)$ and $S(E_2)$.

**Note:** It *is* true that the meridians and parallels of a surface of revolution are always principal, this is *not* true for general orthogonal parametrizations.

*O'Neill*, top of page 235.

**Exercise 6.1** Let $M$ be the surface of revolution obtained by rotating the 'half ellipse'

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad y > 0, \quad a > b > 0$$

about the $x$-axis.

(a) Use the parametrization of the profile curve

$$\alpha(u) = (a\cos u, b\sin u), \quad 0 < u < \pi$$

to calculate the Gaussian curvature function on $M$.

(b) As defined, $M$ is not a closed surface since the points $(\pm a, 0, 0)$ are missing. The parametrization may safely be extended to include these points for reasons that *O'Neill* discussed in an exercise in Chapter IV.

If we do this extension, what are the maximum and minimum values of the Gaussian curvature and where do these extreme values occur?

**Exercise 6.2** Complete the investigation of surfaces of revolution with constant Gaussian curvature by considering the case $K = 0$. The methods used in Examples 6.5 and 6.6 of *O'Neill* can be adapted to this case.

[*Solutions on page 28* ]

# 7 Summary

---

## Read *O'Neill, Chapter V, Section 7, page 244.*

---

We have made the basic definitions for measuring the shape of a surface and developed methods of computing the various functions.

The partial velocities have played a central role in this work. We have made very heavy use of the principle that all forms of directional derivative with respect to the partial velocities reduce to partial differentiation with respect to the parameters $u$ and $v$.

The functions

$$E,\ F,\ G,\ l,\ m,\ n$$

have enabled us to prove the concise formulas

$$K = \frac{ln - m^2}{EG - F^2} \quad \text{and} \quad H = \frac{Gl + En - 2Fm}{2(EG - F^2)}$$

for the Gaussian and mean curvature functions.

The principal curvatures can be calculated as the solutions of the quadratic equation

$$k^2 - 2Hk + K = 0$$

to give

$$k_1, k_2 = H \pm \sqrt{H^2 - K}.$$

We have also discussed the link between the shape operator as a linear transformation on each tangent plane and the various curvature functions, and that the determinant, eigenvectors and eigenvalues of the shape operators all have geometric significance.

# Solutions to the exercises

## Solution 1.1

(a) We calculate the partial derivatives and then use their cross product to define $U$. To save space, we suppress the parameters $u$ and $v$.

$$\begin{aligned}\mathbf{x}_u &= (-r\sin u, r\cos u, 0),\\ \mathbf{x}_v &= (0,0,1),\\ \mathbf{x}_u\times\mathbf{x}_v &= (r\cos u, r\sin u, 0).\\ \|\mathbf{x}_u\times\mathbf{x}_v\|^2 &= r^2(\cos^2 u+\sin^2 u)\\ &= r^2.\\ U(\mathbf{x}(u,v)) &= (\cos u, \sin u, 0).\end{aligned}$$

There is only one other possible choice for $U$, the negative of the one given. However, as defined above, $U$ does point outward from the surface as required.

(b) Applying the result in the commentary, we have

$$\begin{aligned}S(\mathbf{x}_u) &= -\nabla_{\mathbf{x}_u}U\\ &= -\frac{\partial U(\mathbf{x}(u,v))}{\partial u}\\ &= -\frac{\partial}{\partial u}(\cos u, \sin u, 0)\\ &= -(-\sin u, \cos u, 0)\\ &= (\sin u, -\cos u, 0).\end{aligned}$$

Similarly

$$\begin{aligned}S(\mathbf{x}_v) &= -\nabla_{\mathbf{x}_v}U\\ &= -\frac{\partial U(\mathbf{x}(u,v))}{\partial v}\\ &= -\frac{\partial}{\partial v}(\cos u, \sin u, 0)\\ &= (0,0,0).\end{aligned}$$

(c) From the calculations above, we have

$$\begin{aligned}S(\mathbf{x}_u) &= -\frac{1}{r}\mathbf{x}_u + 0\mathbf{x}_v,\\ S(\mathbf{x}_v) &= 0\mathbf{x}_u + 0\mathbf{x}_v.\end{aligned}$$

It follows that the matrix is

$$\begin{pmatrix} -\frac{1}{r} & 0\\ 0 & 0\end{pmatrix}.$$

In this case there is no harm done if you do transpose the matrix!

## Solution 1.2

We follow the same steps as above, beginning with the partial velocities.

$$\begin{aligned}\mathbf{x}_u &= r(-\sin u\cos v, -\sin u\sin v, \cos u),\\ \mathbf{x}_v &= r(-\cos u\sin v, \cos u\cos v, 0),\\ \mathbf{x}_u\times\mathbf{x}_v &= r^2(-\cos^2 u\cos v, -\cos^2 u\sin v, -\sin u\cos u)\\ &= -r^2\cos u(\cos u\cos v, \cos u\sin v, \sin u).\end{aligned}$$

Note that the norm of the cross product is

$$r^2|\cos u|$$

since the vector part above is of unit length.

By comparing this cross product with the parametrization, we see that

$$U(\mathbf{x}(u,v)) = (\cos u\cos v, \cos u\sin v, \sin u)$$

defines an outward unit normal vector field on $\Sigma$.

Next we calculate the values of $S(\mathbf{x}_u)$, etc.

$$\begin{aligned}S(\mathbf{x}_u) &= -\nabla_{\mathbf{x}_u}U\\ &= -\frac{\partial U(\mathbf{x}(u,v))}{\partial u}\\ &= -\frac{\partial}{\partial u}(\cos u\cos v, \cos u\sin v, \sin u)\\ &= -(-\sin u\cos v, -\sin u\sin v, \cos u)\\ &= -\frac{1}{r}\mathbf{x}_u(u,v) + 0\mathbf{x}_v(u,v).\end{aligned}$$

Similarly,

$$\begin{aligned}S(\mathbf{x}_v) &= -\nabla_{\mathbf{x}_v}U\\ &= -\frac{\partial U(\mathbf{x}(u,v))}{\partial v}\\ &= -\frac{\partial}{\partial v}(\cos u\cos v, \cos u\sin v, \sin u)\\ &= -(-\cos u\sin v, \cos u\cos v, 0)\\ &= 0\mathbf{x}_u(u,v) - \frac{1}{r}\mathbf{x}_v(u,v).\end{aligned}$$

It follows that the matrix representing $S$ with respect to the partial velocities is

$$\begin{pmatrix} -\frac{1}{r} & 0\\ 0 & -\frac{1}{r}\end{pmatrix}.$$

Now suppose that the tangent vector $\mathbf{v}$ to $\Sigma$ is given by

$$\mathbf{v} = a\mathbf{x}_u + b\mathbf{x}_v.$$

Then, by linearity,

$$\begin{aligned}S(\mathbf{v}) &= S(a\mathbf{x}_u + b\mathbf{x}_v)\\ &= aS(\mathbf{x}_u) + bS(\mathbf{x}_v)\\ &= a\left(-\frac{1}{r}\right)\mathbf{x}_u + b\left(-\frac{1}{r}\right)\mathbf{x}_v\\ &= -\frac{1}{r}(a\mathbf{x}_u + b\mathbf{x}_v)\\ &= -\frac{\mathbf{v}}{r}.\end{aligned}$$

## Solution 2.1

(a) From Exercise 1 of Section 1 we have that the matrix representing $S$ with respect to the partial velocities is

$$\begin{pmatrix} -\frac{1}{r} & 0\\ 0 & 0\end{pmatrix}.$$

The eigenvalue equation is

$$k^2 + \frac{k}{r} = 0,$$

with solutions

$$k_1 = -\frac{1}{r} \quad\text{and}\quad k_2 = 0.$$

These are the principal curvatures.

It is clear from the matrix that

$$S(\mathbf{x}_u) = k_1\mathbf{x}_u \quad\text{and}\quad S(\mathbf{x}_v) = k_2\mathbf{x}_v.$$

Thus the partial velocities are eigenvectors of $S$ and so they define the principal directions at each point.

(b) Since

$$-\frac{1}{r} \neq 0,$$

there can be no umbilic points on $M$.

(c) Suppose that $\mathbf{u}$ is a unit vector in the required direction. Since

$$\mathbf{x}_u + \mathbf{x}_v = (-r\cos u, r\sin u, 1)$$

and

$$\|\mathbf{x}_u + \mathbf{x}_v\| = \sqrt{1+r^2},$$

we have

$$\begin{aligned} k(\mathbf{u}) &= \left(\frac{1}{\sqrt{1+r^2}}\right)^2 S(\mathbf{x}_u + \mathbf{x}_v)\cdot(\mathbf{x}_u + \mathbf{x}_v) \\ &= \frac{1}{1+r^2}\left(S(\mathbf{x}_u) + S(\mathbf{x}_v)\right)\cdot(\mathbf{x}_u + \mathbf{x}_v) \\ &\quad \text{(linearity of } S\text{)} \\ &= \frac{1}{1+r^2}\left(-\frac{1}{r}\mathbf{x}_u + 0\right)\cdot(\mathbf{x}_u + \mathbf{x}_v) \\ &= -\frac{1}{r(1+r^2)} \\ &\quad \times (-r\sin u, r\cos u, 0)\cdot(-r\sin u, r\cos u, 1) \\ &= -\frac{1}{r(1+r^2)}(r^2) \\ &= -\frac{r}{1+r^2}. \end{aligned}$$

### Solution 2.2

Note that $t = 0$ corresponds to the point

$$\mathbf{x}(0,0)$$

of the cylinder.

We can use the unit normal already found above to deduce that the unit normal to the cylinder at $t = 0$ is

$$U(\mathbf{x}(0,0)) = (1,0,0).$$

The acceleration of the curves can be calculated directly. The velocity and acceleration are

$$\begin{aligned} \alpha'(t) &= (-r\sin t, r\cos t, \pm n t^{n-1}), \\ \alpha''(t) &= (-r\cos t, -r\sin t, \pm n(n-1)t^{n-2}). \end{aligned}$$

Note that, for all such curves,

$$\alpha'(0) = (0, r, 0),$$

so the assertion about their velocities is correct.

We have

$$\alpha''(0) = (-r, 0, 0)$$

if $n > 2$ and

$$\alpha''(0) = (-r, 0, \pm 2)$$

if $n = 2$.

In either case

$$\alpha''(0)\cdot U(\mathbf{x}(0,0)) = -r.$$

This completes the proof of the assertion.

### Solution 3.1

We use the notation of the section of *O'Neill*.

At an umbilic point, we have $k_1 = k_2$. But then

$$K = k_1 k_2 = k_1^2 \geq 0.$$

Hence, if $K < 0$ everywhere on the surface, there can be no umbilic points on the surface.

If $K \leq 0$ on the surface and we know that $K \geq 0$ at an umbilic point, at such a point we must have $K = 0$. But then

$$k_1 = k_2 = 0.$$

Thus any such umbilic point is planar.

### Solution 3.2

The matrix obtained in Section 2 was

$$\begin{pmatrix} \dfrac{-uv}{(1+u^2+v^2)^{3/2}} & \dfrac{1+u^2}{(1+u^2+v^2)^{3/2}} \\[2ex] \dfrac{1+v^2}{(1+u^2+v^2)^{3/2}} & \dfrac{-uv}{(1+u^2+v^2)^{3/2}} \end{pmatrix}.$$

This has determinant

$$\begin{aligned} K &= \frac{u^2v^2 - (1+v^2)(1+u^2)}{(1+u^2+v^2)^3} \\ &= \frac{-(1+u^2+v^2)}{(1+u^2+v^2)^3} \\ &= -\frac{1}{(1+u^2+v^2)^2} \end{aligned}$$

and half the trace is

$$H = \frac{-uv}{(1+u^2+v^2)^{3/2}}.$$

A planar point is one where $k_1 = k_2 = 0$ and hence $K = 0$. However, we have seen that

$$K = \frac{-1}{(1+u^2+v^2)^2} < 0$$

so there can be no planar points.

### Solution 3.3

(a) Let $\mathbf{u}$ and $\mathbf{v}$ be two orthogonal unit tangent vectors at $\mathbf{p}$. In the notation of Corollary 2.6, these correspond to angles of, say, $\vartheta$ and $\vartheta + \pi/2$. Now

$$\begin{aligned} \cos(\vartheta + \pi/2) &= \cos\vartheta\cos(\pi/2) - \sin\vartheta\sin(\pi/2) \\ &= -\sin\vartheta, \\ \sin(\vartheta + \pi/2) &= \sin\vartheta\cos(\pi/2) + \cos\vartheta\sin(\pi/2) \\ &= \cos\vartheta. \end{aligned}$$

Hence

$$\begin{aligned} k(\mathbf{u}) &= k_1\cos^2\vartheta + k_2\sin^2\vartheta, \\ k(\mathbf{v}) &= k_1\sin^2\vartheta + k_2\cos^2\vartheta. \end{aligned}$$

If we add and use $\cos^2\vartheta + \sin^2\vartheta = 1$, we obtain

$$\begin{aligned} k(\mathbf{u}) + k(\mathbf{v}) &= 2k_1 + 2k_2 \\ &= 2H(\mathbf{p}). \end{aligned}$$

The result follows.

Note that the above can also be used to justify the passing remark in the question:

$$\begin{aligned} k(\mathbf{u})k(\mathbf{v}) &= (k_1^2 + k_2^2)\cos^2\vartheta\sin^2\vartheta \\ &\quad + k_1k_2(\cos^4\vartheta + \sin^4\vartheta) \neq K. \end{aligned}$$

(b) We use

$$\cos 2\vartheta = 2\cos^2\vartheta - 1 = 1 - 2\sin^2\vartheta.$$

We have

$$\begin{aligned} \int_0^{2\pi} k(\vartheta)\,d\vartheta &= \int_0^{2\pi}\left(k_1\cos^2\vartheta + k_2\sin^2\vartheta\right)d\vartheta \\ &= \int_0^{2\pi}\left(\frac{k_1}{2}(\cos 2\vartheta + 1) + \frac{k_2}{2}(1 - \cos 2\vartheta)\right)d\vartheta \\ &= \frac{k_1+k_2}{2}\int_0^{2\pi} d\vartheta + \frac{k_1-k_2}{2}\int_0^{2\pi}\cos 2\vartheta\,d\vartheta \\ &= \frac{k_1+k_2}{2}\times 2\pi + \frac{k_1-k_2}{2}\left[\frac{1}{2}\sin 2\vartheta\right]_0^{2\pi} \\ &= 2\pi H(\mathbf{p}). \end{aligned}$$

Thus

$$\frac{1}{2\pi}\int_0^{2\pi} k(\vartheta)\,d\vartheta = H(\mathbf{p}).$$

### Solution 4.1

This question requires the careful application of the various definitions and formulas. Firstly,

$$\mathbf{x}_u = (1, 0, f_u),$$
$$\mathbf{x}_v = (0, 1, f_v).$$

Thus

$$E = 1 + f_u^2,$$
$$F = f_u f_v,$$
$$G = 1 + f_v^2,$$

as asserted.

We can define a unit normal vector field, as usual, by

$$\begin{aligned} U &= \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} \\ &= \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}} \\ &= \frac{(-f_u, -f_v, 1)}{W}. \end{aligned}$$

Next, we need $l$, $m$ and $n$. Now

$$\mathbf{x}_{uu} = (0, 0, f_{uu}),$$
$$\mathbf{x}_{uv} = (0, 0, f_{uv}),$$
$$\mathbf{x}_{vv} = (0, 0, f_{vv}).$$

Hence

$$l = \frac{f_{uu}}{W},$$
$$m = \frac{f_{uv}}{W},$$
$$n = \frac{f_{vv}}{W},$$

as required.

The Gaussian and mean curvatures are calculated as follows.

$$\begin{aligned} K &= \frac{ln - m^2}{EG - F^2} \\ &= \frac{(f_{uu} f_{vv} - f_{uv}^2)/W^2}{(1 + f_u^2)(1 + f_v^2) - f_u^2 f_v^2} \\ &= \frac{f_{uu} f_{vv} - f_{uv}^2}{W^2(1 + f_u^2 + f_v^2)} \\ &= \frac{f_{uu} f_{vv} - f_{uv}^2}{W^4}. \end{aligned}$$

$$\begin{aligned} H &= \frac{Gl + En - 2Fm}{2(EG - F^2)} \\ &= \frac{((1 + f_v^2) f_{uu} + (1 + f_u^2) f_{vv} - 2 f_u f_v f_{uv})/W}{2W^2} \\ &= \frac{(1 + f_v^2) f_{uu} + (1 + f_u^2) f_{vv} - 2 f_u f_v f_{uv}}{2W^3}. \end{aligned}$$

### Solution 4.2

(a) From the previous solution, the surface is flat if, and only if, $K = 0$. This means that the requirement is

$$f_{uu} f_{vv} - f_{uv}^2 = 0.$$

(b) The result follows directly by requiring the numerator of the expression for $H$ to be zero.

### Solution 4.3

You may have worked from first principles or used the results of the first exercise. We do the latter.

Here

$$\begin{aligned} f(u, v) &= \log \cos v - \log \cos u \\ f_u(u, v) &= \frac{\sin u}{\cos u} \\ &= \tan u; \\ f_v(u, v) &= -\frac{\sin v}{\cos v} \\ &= -\tan v; \\ W^2 &= (1 + f_u^2 + f_v^2) \\ &= 1 + \tan^2 u + \tan^2 v. \end{aligned}$$

Hence

$$f_{uu}(u, v) = \sec^2 u,$$
$$f_{uv}(u, v) = 0,$$
$$f_{vv}(u, v) = -\sec^2 v.$$

Substituting in the formulas

$$\begin{aligned} K &= \frac{f_{uu} f_{vv} - f_{uv}^2}{W^4} \\ &= \frac{-\sec^2 u \sec^2 v}{W^4}, \end{aligned}$$

$$\begin{aligned} H &= \frac{(1 + f_v^2) f_{uu} + (1 + f_u^2) f_{vv} - 2 f_u f_v f_{uv}}{2W^3} \\ &= \frac{(1 + \tan^2 v) \sec^2 u - (1 + \tan^2 u) \sec^2 v}{2W^3} \\ &= \frac{\sec^2 v \sec^2 u - \sec^2 u \sec^2 v}{2W^3} \\ &= 0. \end{aligned}$$

Hence $M$ is minimal and $K$ is as stated.

### Solution 5.1

Being an 'if and only if' proof, this solution is in two parts.

First, assume that $\alpha$ is a straight line in $\mathbf{E}^3$. Then

$$\alpha'' = 0.$$

Since $\alpha'' \cdot U = 0$, the acceleration of $\alpha$ is (trivially) normal to the surface $M$ and so $\alpha$ is a geodesic.

Also

$$\alpha'' = 0 \times \mathbf{x}_u + 0 \times \mathbf{x}_v$$

for any patch of $M$ and so the acceleration is also tangent to $M$. Thus $\alpha$ is also asymptotic.

Conversely, suppose that $\alpha$ is geodesic and asymptotic. Then $\alpha''$ is simultaneously tangent to and normal to $M$. Hence

$$\alpha'' = 0$$

and $\alpha$ is a straight line in $\mathbf{E}^3$.

### Solution 5.2

An algebraic approach uses the information assembled, in *O'Neill's* examples about the torus:

$$\begin{aligned} \mathbf{x}_u &= (-r\sin u\cos v, -r\sin u\sin v, r\cos u), \\ \mathbf{x}_v &= (-(R+r\cos u)\sin v, (R+r\cos u)\cos v, 0), \\ U &= (\cos u\cos v, \cos u\sin v, \sin u) \quad \text{(outward normal)}. \end{aligned}$$

(a) This curve is the parameter curve defined by $u = \pi/2$. That is

$$\begin{aligned} \alpha(t) &= \mathbf{x}(\pi/2, t) \\ &= (R\cos t, R\sin t, r). \end{aligned}$$

Thus, on the curve, we have that the velocity is $\mathbf{x}_v(\pi/2, t)$ and so

$$\begin{aligned} \alpha'(t) &= (-R\sin t, R\cos t, 0), \\ U(\alpha(t)) &= (0,0,1). \end{aligned}$$

Hence

$$\begin{aligned} \alpha''(t) &= (-R\cos t, -R\sin t, 0), \\ U' &= (0,0,0). \end{aligned}$$

Using the various tests, we have

$$\begin{aligned} &U' = 0\alpha', \\ &U\cdot\alpha'' = 0, \\ &U \text{ is orthogonal to } \alpha''. \end{aligned}$$

Thus $\alpha$ is principal, from the first test, and the principal curvature is zero along the curve.

From the second test we deduce that $\alpha$ is asymptotic. (This fits with the result that it is principal with principal curvature zero.)

Finally, the last observation shows that $\alpha$ is not geodesic.

**Note:** Geometrically, these results accord with intuition. The unit normal on the curve points upwards from the plane of the curve (which is a circle). The acceleration points towards the centre of the circle and is, therefore, tangent to the surface. Finally, the unit normal is parallel on the curve and so the normal curvature is zero along it. (This last sentence meaning that the normal curvature in the direction of the tangent to the curve is zero.)

(b) This curve is the parameter curve $u = 0$. Thus, as above, we have

$$\begin{aligned} \beta(t) &= \mathbf{x}(0, t), \\ \beta'(t) &= (-(R+r)\sin t, (R+r)\cos t, r), \\ U(\beta(t)) &= (\cos t, \sin t, 0). \end{aligned}$$

Hence

$$\begin{aligned} \beta''(t) &= (-(R+r)\cos t, -(R+r)\sin t, 0), \\ U' &= (-\sin t, \cos t, 0). \end{aligned}$$

Using the various tests, we have

$$\begin{aligned} U' &= \frac{\beta'}{(R+r)}, \\ U\cdot\beta'' &= -(R+r) \neq 0, \\ U &= -\frac{\beta''}{(R+r)}. \end{aligned}$$

Hence $\beta$ is principal, with principal curvature

$$\begin{aligned} \frac{\beta''\cdot U}{\beta'\cdot\beta'} &= \frac{-(R+r)}{(R+r)^2} \\ &= -\frac{1}{(R+r)}. \end{aligned}$$

The curve is not asymptotic, but is geodesic.

(c) We may apply the same sort of techniques as above and in the example in the text. Since

$$\mathbf{x}(u,v) = (u, v, uv),$$

the $x$-axis can be described as the parameter curve $v = 0$, giving

$$\alpha(t) = (t, 0, 0).$$

Since

$$\begin{aligned} \alpha'(t) &= (1,0,0), \\ \alpha''(t) &= (0,0,0), \end{aligned}$$

we can see that the acceleration is (trivially) both tangent to and normal to the surface.

Along $\alpha$, we have

$$U = \frac{(0,-t,1)}{\sqrt{1+t^2}},$$

and so

$$U' = \frac{(0,-1,-t)}{(1+t^2)^{3/2}}.$$

Since $U'\cdot\alpha' = 0$, $U'$ and $\alpha'$ are orthogonal and so the curve is asymptotic. Also, since $U'$ and $\alpha'$ are not collinear, so $\alpha$ is not principal. Since the acceleration is (trivially) a multiple of the unit normal, the curve is geodesic.

### Solution 5.3

We assemble a few preliminary facts. From the Frenet formulas, we have that $\alpha' = T$ and so

$$\alpha'' = T' = \kappa N.$$

(a) Since $\alpha$ is asymptotic, we know that its acceleration is tangent to $M$. Hence, from the remarks above, $N$ is tangent to $M$.

But, since $\alpha$ is a curve in $M$, its velocity $T$ is also tangent to $M$. Hence

$$B = T\times N$$

is normal to $M$.

(b) Applying the result that, along a curve, the shape operator is given by $-U'$, we have

$$S(T) = -B' = -(-\tau N) = \tau N,$$

by using the Frenet formula for $B'$.

### Solution 6.1

We can apply the general discussion of surfaces of revolution in *O'Neill* to this case.

If we set

$$\begin{aligned} g(u) &= a\cos u, \\ h(u) &= b\sin u, \end{aligned}$$

then we obtain the parametrization

$$\mathbf{x}(u,v) = (a\cos u, b\sin u\cos v, b\sin u\sin v) \quad 0 < u < \pi$$

of $M$.

(a) We have

$$\begin{aligned} g'(u) &= -a\sin u, \\ g''(u) &= -a\cos u, \\ h'(u) &= b\cos u, \\ h''(u) &= -b\sin u, \\ g'^2 + h'^2 &= a^2\sin^2 u + b^2\cos^2 u, \\ g'h'' - g''h' &= ab(\sin^2 u + \cos^2 u) \\ &= ab. \end{aligned}$$

Hence

$$\begin{aligned} K &= \frac{-g'(g'h'' - g''h')}{h(g'^2 + h'^2)^2} \\ &= \frac{a^2 b \sin u}{b\sin u(a^2\sin^2 u + b^2\cos^2 u)^2} \\ &= \frac{a^2}{(a^2\sin^2 u + b^2\cos^2 u)^2}. \end{aligned}$$

(b) From the formula for $K$ above, we can see that $K$ is a maximum when the denominator is a minimum and vice versa. It is slightly easier to find the extreme values of the denominator if we express it in terms of $\cos u$ only.

Since

$$\begin{aligned} a^2\cos^2 u + b^2\sin^2 u &= a^2\cos^2 u + b^2(1 - \cos^2 u) \\ &= b^2 + (a^2 - b^2)\cos^2 u, \end{aligned}$$

the maximum value of the denominator occurs when $u = 0$ or $u = \pi$ and the minimum when $u = \pi/2$.

Thus the maximum Gaussian curvature is

$$K(\mathbf{x}(0,v)) = K(\mathbf{x}(\pi,v)) = \frac{a^2}{b^4},$$

and occurs for all points with $u = 0$, i.e. at $(\pm a, 0, 0)$.

The minimum Gaussian curvature is

$$K(\mathbf{x}(\pi/2,v)) = \frac{a^2}{a^4} = \frac{1}{a^2}$$

and occurs for all points on the meridian $u = \pi/2$.

## Solution 6.2

If we assume that the surface has been parametrized in such a way that the profile curve is unit speed, that is using the 'canonical' parametrization, we have

$$K = \frac{-h''}{h}.$$

We must, therefore, solve $K = 0$, that is

$$-h'' = 0,$$

subject to the conditions $h > 0$ and $\|h'\| < 1$.

The general solution of $h'' = 0$ is

$$h(t) = at + b,$$

where $a$ and $b$ are constants.

Since $h'(t) = a$, the condition $\|h'\| < 1$ requires

$$\|a\| < 1.$$

Next, we apply the formula from page 239 to obtain $g$.

$$\begin{aligned} g(u) &= \int_0^u \sqrt{1 - (h'(t))^2}\,dt \\ &= \int_0^u \sqrt{1 - a^2}\,dt \\ &= \left[\sqrt{(1 - a^2)}\,t\right]_0^u \\ &= u\sqrt{1 - a^2}. \end{aligned}$$

Thus, the profile curve is

$$\alpha(t) = (u\sqrt{1 - a^2}, au + b).$$

We can reparametrize this by replacing $u$ by

$$\frac{u}{\sqrt{1 - a^2}}$$

to get a profile curve of the form

$$(u, cu + d),$$

where $c$ and $d$ are constants.

Thus the profile curves are straight lines and the surfaces are parts of cones (if the slope $c$ is not zero) or parts of cylinders (if $c = 0$).

ISBN 0 7492 4774 6